Homework 6

Math 253, Spring 2021

Name: Key

- This packet is due at **12:00 pm PDT (noon) on Friday, May 14**.
- Neatly fill out this packet either directly or on your own paper. Then upload your answers to Canvas. Please clearly number each problem and keep them in order.
- For each problem in this packet, **organize and show all of your work and justification** for your answer.
- Make sure that it is clear what your final answer is. This may mean boxing/circling it or it may mean doing neither.
- If you are required to draw a graph of a function, each axis must be labeled and given a scale.
- I will not be grading every individual problem, but answers will be available on Canvas after the due date.
- I am happy to help you however I can. Here are some options:
  - Come to my office hours (see syllabus).
  - Email me directly at gknapp4@uoregon.edu

1. Find the radius and interval of convergence for the following power series.

(a) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$

Want: $1 > \lim_{n\to\infty} \left| \frac{(2x)^{n+1}}{n+1} \cdot \frac{n}{(2x)^n} \right|$

$= \lim_{n\to\infty} \left| 2x \cdot \frac{n}{n+1} \right| = |2x|$

$1 > |2x| \iff \frac{1}{2} > |x| \iff x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

$\boxed{R = \frac{1}{2}}$

Check: $x = \frac{1}{2}$

$\sum_{n=1}^{\infty} \frac{(2\cdot\frac{1}{2})^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test ($p = 1$)

Check: $x = -\frac{1}{2}$

$\sum_{n=1}^{\infty} \frac{(2\cdot-\frac{1}{2})^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ diverges by the A.S.T.

$\boxed{I = \left[-\frac{1}{2}, 1\right)}$

(b) $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{\sqrt{n}}$

Goal: $1 > \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\cdot x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-1)^n \cdot x^n} \right|$

$= \lim_{n\to\infty} \left| (-1)\cdot x \cdot \sqrt{\frac{n}{n+1}} \right|$

$= |x|$

$1 > |x| \iff x \in (-1, 1)$

$\boxed{R = 1}$

Check: $x = -1$

$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n}$ (even)

$= \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test ($p = 1$)

Check: $x = 1$

$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the A.S.T.

$\boxed{I = (-1, 1]}$

(c) $\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$

Goal: $1 > \lim_{n\to\infty} \left| \frac{10^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n \cdot x^n} \right|$

$= \lim_{n\to\infty} \left| \frac{10x}{n+1} \right| = 0$

$1 > 0$ is true for all x, so

$\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$ converges for all $x \in (-\infty, \infty) = I$

$\infty = R$

(d) $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{\ln(2n)}$

Goal: $1 > \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\cdot x^{n+1}}{\ln(2(n+1))} \cdot \frac{\ln(2n)}{(-1)^n x^n} \right|$

$= \lim_{n\to\infty} \left| (-1)\cdot x \cdot \frac{\ln(2n)}{\ln(2n+2)} \right|$

Note: $\lim_{n\to\infty} \frac{\ln(2n)}{\ln(2n+2)} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{\frac{1}{2n}\cdot 2}{\frac{1}{2n+2}\cdot 2}$

$= \lim_{n\to\infty} \frac{1/n}{1/n+1}$

$= \lim_{n\to\infty} \frac{n+1}{n} = 1$

So $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}\cdot x^{n+1}}{\ln(2(n+1))} \cdot \frac{\ln(2n)}{(-1)^n x^n} \right| = |x|$

$1 > |x| \iff x \in (-1,1) \implies \boxed{R = 1}$

Check: $x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n\cdot(-1)^n}{\ln(2n)} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\ln(2n)} = \sum_{n=1}^{\infty} \frac{1}{\ln(2n)}$

$\frac{1}{\ln(2n)} \geq \frac{1}{2n}$, so by the D.C.T. $\sum_{n=1}^{\infty} \frac{1}{\ln(2n)}$ diverges

Check: $x = 1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(2n)}$ converges by the A.S.T.

$\boxed{I = (-1, 1]}$

2. Find the **radius** of convergence for each of the following series. You do not need to find the actual interval.

(a) $\sum_{n=1}^{\infty} \frac{n!(x+5)^n}{n^n}$

Want: $1 > \lim_{n\to\infty} \left| \frac{(n+1)!\cdot(x+5)^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!\,(x+5)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \cdot \frac{(x+5)^{n+1}}{(x+5)^n} \right|$

$= \lim_{n\to\infty} \left| (n+1) \cdot \frac{n^n}{(n+1)^{n+1}} \cdot (x+5) \right|$

$= \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^n} \cdot (x+5) \right|$

$= \lim_{n\to\infty} \left| \left(\frac{n}{n+1}\right)^n \cdot (x+5) \right|$

$= \lim_{n\to\infty} \left| \frac{1}{\left(\frac{n+1}{n}\right)^n} \cdot (x+5) \right|$

$= \lim_{n\to\infty} \left| \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot (x+5) \right|$

$= \frac{|x+5|}{e}$

$1 > \frac{|x+5|}{e} \iff e > |x+5| \Rightarrow \boxed{R = e}$

(b) $\sum_{n=1}^{\infty} \frac{(n!)^3(2x-4)^n}{(3n)!}$

Goal: $1 > \lim_{n\to\infty} \left| \frac{((n+1)!)^3 \cdot (2x-4)^{n+1}}{(3(n+1))!} \cdot \frac{(3n)!}{(n!)^3 \cdot (2x-4)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{((n+1)!)^3}{(n!)^3} \cdot \frac{(3n)!}{(3n+3)!} \cdot \frac{(2x-4)^{n+1}}{(2x-4)^n} \right|$

$= \lim_{n\to\infty} \left| \left(\frac{(n+1)!}{n!}\right)^3 \cdot \frac{(3n)(3n-1)\cdots 2\cdot 1}{(3n+3)(3n+2)(3n+1)(3n)(3n-1)\cdots 2\cdot 1} \cdot (2x-4) \right|$

$= \lim_{n\to\infty} \left| (n+1)^3 \cdot \frac{1}{(3n+3)(3n+2)(3n+1)} \cdot 2(x-2) \right|$

$= \lim_{n\to\infty} \left| \frac{n^3 + \text{L.D.T.}}{27n^3 + \text{L.D.T.}} \cdot 2(x-2) \right| = \frac{2}{27} \cdot |x-2|$

$1 > \frac{2}{27} \cdot |x-2| \iff \frac{27}{2} > |x-2|$

$\Rightarrow \boxed{R = \frac{27}{2}}$

3. Find the power series for each function with the given center a, and identify its interval of convergence.

(a) $f(x) = \dfrac{x}{1+x^2}$, $a = 0$

$$f(x) = x \cdot \frac{1}{1-(-x^2)} = x \cdot \sum_{n=0}^{\infty} (-x^2)^n$$

$$= x \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}$$

converges if and only if $|-x^2| < 1$

$\Updownarrow$

$x^2 < 1$

$\Updownarrow$

$|x| < 1$

$$\Rightarrow \boxed{I = (-1, 1)}$$

(b) $f(x) = \frac{1}{2-x}$, $a = 7$

$$f(x) = \frac{1}{2-x} = \frac{1}{2-(x-7+7)} = \frac{1}{2-(x-7)-7}$$

$$= \frac{1}{-5-(x-7)} = \frac{1}{-5} \cdot \frac{1}{1-\left(\frac{x-7}{-5}\right)}$$

$$= -\frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{x-7}{-5}\right)^n$$

$$= -\frac{1}{5} \sum_{n=0}^{\infty} \frac{(x-7)^n}{(-5)^n} = \sum_{n=0}^{\infty} \frac{(x-7)^n}{(-5)^{n+1}}$$

converges if and only if

$$\left|\frac{x-7}{-5}\right| < 1 \iff \frac{|x-7|}{5} < 1 \iff |x-7| < 5$$

$$\iff x \in (2, 12)$$

$$\Rightarrow \boxed{I = (2, 12)}$$